University of California, Berkeley
Physics 110B, Fall 2004 (*Strovink*)

PROBLEM SET 5

Due at 5 PM on Monday, September 27, 2004

Problems 21-25 complete our study of Special Relativity, except for motion of relativistic particles in EM fields.

21. (Effect of inefficient rocket engine)
All of the energy put out by the rocket engine depicted in SCSR Fig. 9 consists of particles emitted straight out the back. Consider the more realistic case in which only a fraction ϵ of the energy output consists of such particles; as seen in a Lorentz frame comoving with the rocket, the balance of the energy is emitted isotropically, for example as thermal photons. Therefore ϵ is the engine's *efficiency*. Otherwise adopt the conditions of SCSR Fig. 9 and carry out a derivation analogous to that found in SCSR §14. Assuming that it starts from rest, show that the final boost of the rocket is reduced directly by this efficiency factor:

$$\eta_{\rm final} = \epsilon\,|\vec{\beta}_1|\ln\frac{m_0}{m_{\rm final}}\ .$$

22. (Wave aberration)
Please refer to SCSR Fig. 10. Consider Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$, with spatial origins coincident at $t = t' = 0$. As usual, frame $\mathcal{S}'$ moves in the $\hat{x} = \hat{x}'$ direction with velocity $\beta_0 c$ relative to frame $\mathcal{S}$. A wave is emitted by a source that is at rest with respect to $\mathcal{S}'$. As seen by an observer in the lab frame $\mathcal{S}$, the wave travels with phase velocity $\beta_{\rm ph}c$ at an angle θ with respect to the $\hat{x}$ direction ($\theta = 0$ if directly approaching, $\theta = \pi$ if directly receding). However, as seen by an observer who is at rest with respect to the frame $\mathcal{S}'$, show that the wave makes a different angle θ' with respect to the $\hat{x}'$ direction, where

$$\tan\theta' = \frac{\sin\theta}{\gamma_0(\cos\theta - \beta_0\beta_{\rm ph})}\ .$$

23. (Lorentz transformation of EM fields)
Consider Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$, with frame $\mathcal{S}'$ moving in the $\hat{x} = \hat{x}'$ direction with velocity $\beta_0 c$ relative to frame $\mathcal{S}$. Using the Lorentz transformation for the field strength tensor,

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F^{\rho\sigma}\ ,$$

and considering explicitly the values of the elements of $F^{\mu\nu}$, as given by SCSR Eq. (62), show that

$$E'_y = \gamma_0(E_y - \beta_0 c B_z)\ ,$$

as claimed by Griffiths' Eq. (12.102).

24. (Relativistic electron-positron beams)
In a straight channel oriented along the $\hat{z}$ axis there are two opposing beams:

- a beam of positrons (charge $+e$) with velocity $+\hat{z}\beta_0 c$.
- a beam of electrons (charge $-e$) with velocity $-\hat{z}\beta_0 c$.

Each beam is confined to a small cylindrical volume of cross sectional area A centered on the $\hat{z}$ axis. Within that volume, there is a uniform number density $= n$ positrons/m^3 and n electrons/m^3.

(a.)
In terms of n, A, e, and β_0, calculate the total current I in the channel due to the sum of both beams (note $I \neq 0$).

(b.)
Use Ampère's Law to calculate the (azimuthal) magnetic field $\vec{B}$ outside the channel a distance s from the $\hat{z}$ axis.

Consider now a Lorentz frame $\mathcal{S}'$ traveling in the $\hat{z}$ direction with velocity $\beta_0 c$ relative to the lab frame described above. (This β_0 is the same β_0 as above.)

(c.)
As seen in $\mathcal{S}'$, calculate the number density n'_+ of *positrons* within the cylindrical volume. (You may use elementary arguments involving

space contraction, or you may use the fact that $(c\rho, \vec{J})$ is a 4-vector, where ρ is the charge density (coul/m^3) and $\vec{J}$ is the current density (amps/m^2).)

(d.)

As seen in $\mathcal{S}'$, calculate the number density n'_- of *electrons* within the cylindrical volume.

(e.)

Calculate the (cylindrically radial) electric field $\vec{E}'$ seen outside the channel in $\mathcal{S}'$. Do this both

- by using the results of (c.) and (d.) plus Gauss's law, and
- by using the results of (b.) plus the rules for relativistic $\vec{E}$ and $\vec{B}$ field transformations.

25.

(a.)

Express $\mu_0 J^\nu$ as the four-divergence of the field strength tensor $F^{\mu\nu}$. Exploiting the antisymmetry of $F^{\mu\nu}$ under interchange of its indices, prove without reference to the specific values of the elements of F that

$$\partial_\mu J^\mu = 0$$

and thus that electric charge must be conserved. (The basic structure of Maxwell's equations would have to be completely reformulated if even the tiniest violation of electric charge conservation were to be observed anywhere in the universe.)

(b.)

Define

$$\epsilon_{\mu\nu\rho\sigma} \equiv g_{\mu\alpha} g_{\nu\beta} g_{\rho\kappa} g_{\sigma\lambda} \epsilon^{\alpha\beta\kappa\lambda} ,$$

where ϵ is as defined after SCSR Eq. (64). Prove that

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma} .$$

(c.)

Without making reference to the specific values of the dual field strength tensor

$$G^{\mu\nu} \equiv \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} ,$$

using both the antisymmetry of F and the antisymmetry of ϵ, prove that

$$\partial_\mu G^{\mu\nu} = 0 .$$

(This is equivalent to the sourceless Maxwell equations.)